THE MINIMUM OF A DEFINITE INTEGRAL FOR UNILATERAL VARIATIONS IN SPACE*

BY

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For the problem of the calculus of variations with fixed end points, the curves which minimize an integral of the form

$$J = \int F(x, y, z, x', y', z')\, ds$$

have previously been studied, provided that the minimizing arc along which the integral J is taken is free to traverse any portion of space or restricted to lie entirely within a given region R. It may happen, however, that while no extremal joining the two fixed end points exists, which is entirely interior to a region R, yet there may be a minimizing curve consisting partly of arcs interior to the region and partly of arcs along its boundary. This gives rise to the problem of investigating the properties which characterize a minimizing

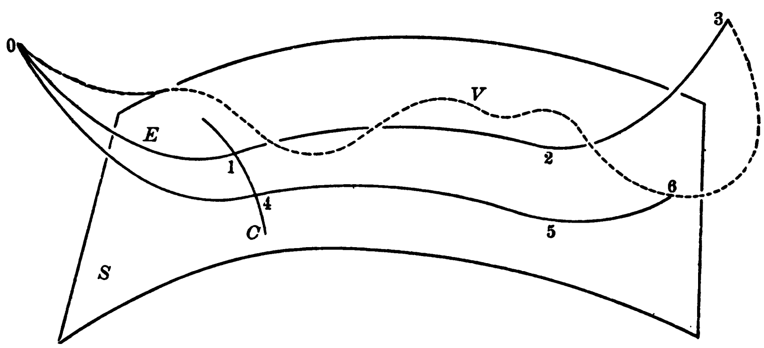

FIG. 1.

curve lying partly on a given surface S, while the variations with which it is compared are restricted to lie on one side of the surface, a problem analogous to one in the plane which has been discussed by Bliss.† For this problem in space a number of necessary conditions have been derived by other writers, but the theory of the analogue of Jacobi's condition is still incomplete.

* Presented to the Society, December 27, 1913.

† These Transactions, vol. 5 (1904), pp. 477–492.

Suppose that the arc E, whose minimizing properties are to be investigated, consists of three parts E_{01}, E_{12}, and E_{23}, as shown in the accompanying figure (Fig. 1). The arc E_{12} must be an extremal of the integral J on the surface S, while the other two, lying on the same side of S and joining the end points of E_{12} to the two fixed points 0 and 3, are necessarily extremals in space and tangent to the arc E_{12}. For the corresponding situation in the plane, the analogue of Jacobi's condition requires only that Jacobi's condition itself be satisfied by each of the arcs E_{01} and E_{23}. For the problem here considered, however, there is in general more required than this. It is not sufficient, in order that E shall minimize the integral, that no conjugate point to 0 lies on the extremal arc E_{01}, and no conjugate point to 2 on the arc E_{23}.*

It will be shown in the following pages that from the two parameter family of space extremals passing through the point 0, a set containing E_{01} and depending upon a single parameter may be selected, each curve of which is tangent to the surface S at a point of a curve C through the point 1. Through the points of C a one parameter family of extremals on S can be passed, each one of which is tangent at a point of C to one of the space extremals through the point 0. Again at any point of one of the single infinitude of surface extremals so determined, there is a space extremal tangent to the surface extremal and lying on the required side of the surface. These form a two parameter family of space extremals tangent to S.

Two new conditions, analogues of the Jacobi condition in the plane case, will be established; (1) the arc E_{12} must not contain a point of contact with the enveloping curve of the one-parameter family of surface extremals above mentioned; and (2) the arc E_{23} must not contain a contact point with the enveloping surface of the two parameter family of space extremals constructed as above tangent to the surface S.

The necessary conditions hitherto derived and the two analogues to the Jacobi condition described in the preceeding paragraph, when strengthened in a manner customary in the calculus of variations, are sufficient to insure that the curve E actually minimizes the integral J.

1. Necessary Conditions Hitherto Derived

Let the curve E joining the points 0 and 3, whose minimizing properties are to be studied, have the equations

$$(E) \qquad x = x(s), \quad y = y(s), \quad z = z(s) \qquad (s_0 \leqq s \leqq s_3),$$

where s is the length of arc measured from the point 0. The curve is supposed

* See a remark by Hadamard in his *Leçons sur le calcul des variations*, p. 457.

not to intersect itself and is made up of three pieces, each of class C'',* as shown in Fig. 1.

The surface S, on which E_{12} lies, has equations of the form

$$(S) \qquad x = \xi(\alpha, \beta), \qquad y = \eta(\alpha, \beta), \qquad z = \zeta(\alpha, \beta).$$

It has no points in common with E_{01} and E_{23} except 1 and 2, and is further supposed to have no multiple or singular points and to be of class C''' in the vicinity of E_{12}.

The function F in the integral J is assumed to have the following properties in a region R of points (x, y, z, x', y', z') for which (x, y, z) ranges over a neighborhood of E, while (x', y', z') are any values not all zero:

$$(A) \qquad F \text{ is of class } C''';$$

$$(B) \qquad F(x, y, z, \kappa x', \kappa y', \kappa z') = \kappa F(x, y, z, x', y', z') \qquad (\kappa > 0);$$

$$(C) \qquad F_1 \neq 0.$$

The property (B) is the usual homogeneity property of the function F which makes the value of the integral independent of the parametric representation of the curve along which it is taken, while (C) specifies that the problem is a regular one.†

If E is to minimize J, then it must satisfy a number of necessary conditions which are already known and which may be stated as follows:

(I) The arcs E_{01} and E_{23} must be space extremals satisfying the differential equations

$$(1) \qquad J^{(x)} = F_x - \frac{d}{ds}F_{x'} = 0, \qquad J^{(y)} = F_y - \frac{d}{ds}F_{y'} = 0, \qquad J^{(z)} = F_z - \frac{d}{ds}F_{z'} = 0;$$

(II) Along the arc E_{12} the equations

$$(2) \qquad \frac{J^{(x)}}{X} = \frac{J^{(y)}}{Y} = \frac{J^{(z)}}{Z} = \lambda(s)$$

must be satisfied, where X, Y, Z are the direction cosines of the normal of the surface S in the direction toward the side of S on which E_{01} and E_{23} lie, while $\lambda(s)$ is merely the notation for the function of s defined by the three ratios;

* An arc is said to be of class $C^{(n)}$ if the derivatives of the functions defining it up to and including those of order n are continuous, and if the expression $\sqrt{x'^2 + y'^2 + z'^2}$ is everywhere different from zero along it.

† For the definition of F_1 and a "regular problem" see a paper by Mason and Bliss, these Transactions, vol. 9 (1908), p. 444. This paper will be referred to hereafter as M. and B.

(III) Along E_{12} the inequality $\lambda(s) \gtreqless 0$ must hold;

(IV) The function

$$E(x, y, z;\ x', y', z';\ p, q, r) = F(x, y, z, p, q, r) - pF_{x'}(x, y, z, x', y', z') - qF_{y'}(x, y, z, x', y', z') - rF_{z'}(x, y, z, x', y', z')$$

must be positive or zero for all values (x, y, z, x', y', z') on the arc E_{03} and for arbitrary values $(p, q, r) \neq (0, 0, 0)$;

(V) At the points 1 and 2, E_{01} and E_{23} must be tangent to the arc E_{12};

(VI) The arc E_{01} must not contain a point conjugate to 0 between 0 and 1.*

From the property IV it follows that the quadratic form

$$Q(x, y, z;\ x', y', z';\ p, q, r) = F_{x'x'}p^2 + F_{y'y'}q^2 + F_{z'z'}r^2 + 2F_{y'z'}qr + 2F_{z'x'}rp + 2F_{x'y'}pq$$

must be positive or zero for the arguments described for E.† The sign of F_1 must therefore be positive, and it can be shown that $Q \geqq 0$ for all values (x, y, z, x', y', z') in the region R and for arbitrary directions (p, q, r).‡ Furthermore Q vanishes only when the directions x', y', z' and p, q, r are in the same line.§ The same properties hold for the E-function, with the further restriction that E vanishes only when x', y', z' and p, q, r are identical (not opposite) directions.||

The condition II can also be stated in terms of the integrand

$$f(\alpha, \beta, \alpha', \beta') = F(\xi, \eta, \zeta, \xi_\alpha \alpha' + \xi_\beta \beta', \eta_\alpha \alpha' + \eta_\beta \beta', \zeta_\alpha \alpha' + \zeta_\beta \beta')$$

of the integral

$$J = \int f(\alpha, \beta, \alpha', \beta')\, ds \qquad (3)$$

* For the conditions (I), (IV), (VI) on the arcs E_{01} and E_{23}, see M. and B., loc. cit., pages 443, 455, 453, respectively. For (II), (III), (V), see Hadamard, loc. cit., pages 133, 179, 456. There is an exceptional case to (VI) mentioned in M. and B., p. 453, which can, however, be treated by a consideration of the second variation instead of the geometric proof used by them. The proof that the condition (IV) must hold along the arc E_{12} can be made in a manner analogous to that used for the other arcs. It is somewhat more difficult because the variations are restricted to lie on one side of the surface S.

† M. and B., p. 455.

‡ The argument is like that of M. and B., p. 456, for the functions $F_{x'x'}, F_{y'y'}, F_{z'z'}$.

§ M. and B., p. 456, formulas (40).

|| The proof of this statement may be made by means of the formulas in M. and B., but more elegantly by the use of the results of Behaghel, Mathematische Annalen, vol. 73 (1913), p. 596. The functions A, B, C used by him may become infinite for directions parallel to the axes and the validity of his proof is questionable in such cases, though his result is correct. The justification for these statements will appear in the near future in a paper by Bliss.

taken along curves on the surface S. Along E_{12} the parameters α and β are functions

$$\alpha = \alpha(s), \qquad \beta = \beta(s) \qquad (s_1 \leqq s \leqq s_2)$$

which are of class C''. After an easy calculation the Euler equations for the integral (3) are seen to have the form

$$(4) \qquad \begin{aligned} f_\alpha - \frac{d}{ds} f_{\alpha'} &= J^{(x)} \xi_\alpha + J^{(y)} \eta_\alpha + J^{(z)} \zeta_\alpha = 0, \\ f_\beta - \frac{d}{ds} f_{\beta'} &= J^{(x)} \xi_\beta + J^{(y)} \eta_\beta + J^{(z)} \zeta_\beta = 0, \end{aligned}$$

and these are satisfied along E_{12} on account of the condition II.

Furthermore the function*

$$f_1 = \frac{f_{\alpha'\alpha'}}{\beta'^2} = -\frac{f_{\alpha'\beta'}}{\alpha'\beta'} = \frac{f_{\beta'\beta'}}{\alpha'^2}$$

has the values

$$f_1 = \frac{Q(x, y, z;\ x', y', z';\ \xi_\alpha, \eta_\alpha, \zeta_\alpha)}{\beta'^2} = \frac{Q(x, y, z;\ x', y', z';\ \xi_\beta, \eta_\beta, \zeta_\beta)}{\alpha'^2},$$

where

$$x' = \xi_\alpha \alpha' + \xi_\beta \beta', \qquad y' = \eta_\alpha \alpha' + \eta_\beta \beta', \qquad z' = \zeta_\alpha \alpha' + \zeta_\beta \beta'.$$

One of these expressions is always well defined and different from zero, since the derivatives x', y', z' can coincide with at most one of the coördinate directions on the surface. The problem of minimizing the integral (3) on the surface is therefore a regular one, and E_{12} is one of its extremals.

If a space extremal satisfying the condition I is tangent to a surface extremal satisfying II, then at the point of contact there is a relation between their curvatures which will be of service later. Let the elements of the second order on the two extremals at the point of contact be, respectively,

$$(x, y, z, x', y', z', \bar{x}'', \bar{y}'', \bar{z}'')$$

and

$$(x, y, z, x', y', z', x'', y'', z'').$$

From equations (1) and (2) it follows that

$$(5) \qquad \lambda X = J^{(x)} - \bar{J}^{(x)}, \qquad \lambda Y = J^{(y)} - \bar{J}^{(y)}, \qquad \lambda Z = J^{(z)} - \bar{J}^{(z)},$$

and

$$\begin{aligned} \lambda\{X(x'' - \bar{x}'') + Y(y'' - \bar{y}'') + Z(z'' - \bar{z}'')\} &= (x'' - \bar{x}'')(J^{(x)} - \bar{J}^{(x)}) + (y'' - \bar{y}'')(J^{(y)} - \bar{J}^{(y)}) \\ &\qquad + (z'' - \bar{z}'')(J^{(z)} - \bar{J}^{(z)}) \\ &= Q(x, y, z;\ x', y', z';\ x'' - \bar{x}'', y'' - \bar{y}'', z'' - \bar{z}''). \end{aligned}$$

* See Bolza, *Vorlesungen über Variationsrechnung*, p. 196.

This last expression can vanish, however, only when

$$x'' - \bar{x}'' = \kappa x', \qquad y'' - \bar{y}'' = \kappa y', \qquad z'' - \bar{z}'' = \kappa z',$$

where κ is a value which may be zero. In case it does vanish the equations (5) show that $\lambda = 0$, as a result of the homogeneity properties of the function F.* Hence the following lemma is true:

LEMMA 1: *If* $\lambda \neq 0$ *at the point of contact of a surface extremal with a space extremal, then at that point the inequality*

$$Xx'' + Yy'' + Zz'' \neq X\bar{x}'' + Y\bar{y}'' + Z\bar{z}''$$

holds; or, in other words, the projections of the first curvatures, $1/r$ *and* $1/\bar{r}$, *of the two extremals on the normal to the surface are different.*

From this point in the paper it will be assumed that the arc E_{03} satisfies the conditions I, II, IV, V and the modified conditions

(III′) $\lambda(s) > 0$ along E_{12};

(VI′) the arc E_{01} does not contain a point conjugate to 0 between 0 and 1 or at 1.†

On the basis of these assumptions further conditions, analogous to Jacobi's conditions for the usual form of the problem, will be derived.

2. The Three Families of Extremals Associated with E_{03}

The totality of solutions of the differential equations (1) passing through 0 contains two arbitrary parameters. As a result of the hypotheses made upon the function F they may be represented in the form

$$(6)\quad x = \phi(s, u, v, w), \qquad y = \psi(s, u, v, w), \qquad z = \chi(s, u, v, w),$$
$$u^2 + v^2 + w^2 = 1,$$

the parameters u, v, w being the direction cosines of the positive tangent to the extremal at the point 0. The particular extremal E_{01} is contained in this set for $u = u_0$, $v = v_0$, $w = w_0$, and the functions $\phi, \psi, \chi, \phi_s, \psi_s, \chi_s$ are of class C' in neighborhoods of the two sets of parameter values

$$s = 0, \qquad u^2 + v^2 + w^2 = 1;$$

$$0 \leqq s \leqq s_1, \qquad u = u_0, \qquad v = v_0, \qquad w = w_0$$

of which the second set defines the curve E_{01}. At the point 0, where $s = 0$, the following identities in u, v, w hold

$$x_0 = \phi(0, u, v, w), \qquad y_0 = \psi(0, u, v, w), \qquad z_0 = \chi(0, u, v, w).$$

* M. and B., p. 441, formula (5).

† The condition VI′ is necessary except in one singular case. See M. and B., p. 453.

Let the notation $\Delta(s, u, v, w)$ designate the determinant

$$\Delta(s, u, v, w) = \begin{vmatrix} \phi_s & \phi_u & \phi_v & \phi_w \\ \psi_s & \psi_u & \psi_v & \psi_w \\ \chi_s & \chi_u & \chi_v & \chi_w \\ 0 & u & v & w \end{vmatrix}.$$

On account of the condition VI′ it follows* that

$$\text{(7)} \qquad \Delta(s, u_0, v_0, w_0) \neq 0 \qquad (0 < s \leqq s_1).$$

The equations

$$\xi(\alpha, \beta) + \omega X = \phi(s, u, v, w),$$
$$\eta(\alpha, \beta) + \omega Y = \psi(s, u, v, w),$$
$$\zeta(\alpha, \beta) + \omega Z = \chi(s, u, v, w),$$

have an initial solution $(\alpha, \beta, \omega; s, u, v, w) = (\alpha_1, \beta_1, 0; s_1, u_0, v_0, w_0)$ corresponding to the point 1, at which the functional determinant for α, β, ω is different from zero. Hence these equations have solutions

$$\text{(8)} \quad \alpha = \alpha(s, u, v, w), \qquad \beta = \beta(s, u, v, w), \qquad \omega = \omega(s, u, v, w)$$

of class C'' near (s_1, u_0, v_0, w_0) and reducing to $(\alpha_1, \beta_1, 0)$ at these values. The derivatives of ω at the point 1 are readily calculated to be

$$\text{(9)} \quad \begin{aligned} \omega_s &= X\phi_s + Y\psi_s + Z\chi_s, \qquad \omega_u = X\phi_u + Y\psi_u + Z\chi_u, \\ \omega_v &= X\phi_v + Y\psi_v + Z\chi_v, \qquad \omega_w = X\phi_w + Y\psi_w + Z\chi_w, \\ \omega_{ss} &= X\phi_{ss} + Y\psi_{ss} + Z\chi_{ss} - L\alpha_s^2 - 2M\alpha_s\beta_s - N\beta_s^2\dagger \\ &= Xx'' + Yy'' + Zz'' - (X\bar{x}'' + Y\bar{y}'' + Z\bar{z}''), \end{aligned}$$

where the notations in the last line are the same as those of the preceding section. The set (x'', y'', z'') belongs to E_{01} at the point 1, and the set $(\bar{x}'', \bar{y}'', \bar{z}'')$ to E_{12}.

The solutions (s, u, v, w) of the equations

$$\text{(10)} \qquad \omega(s, u, v, w) = 0, \qquad \omega_s(s, u, v, w) = 0, \qquad u^2 + v^2 + w^2 = 1$$

determine points where the extremal (6) is tangent to the surface S. On account of the tangency of E_{01} and E_{12}, it is evident that among the solutions are the values (s_1, u_0, v_0, w_0). On the other hand the determinants of the matrix

$$\left\| \begin{matrix} \omega_{ss} & \omega_{su} & \omega_{sv} & \omega_{sw} \\ \omega_s & \omega_u & \omega_v & \omega_w \\ 0 & u & v & w \end{matrix} \right\|$$

* See M. and B., §3 and p. 453.

† For the definitions of L, M, N and their relations to the first curvature, see Scheffers, *Anwendung der Differential und Integral-Rechnung auf Geometrie*, vol. 2, page 494 (D).

cannot all vanish at (s_1, u_0, v_0, w_0). At these values $\omega_s = 0$ since E_{01} is tangent to the surface S; but from equations (9) and the property (7) it follows readily that the determinants of the last two rows can not all vanish; and from III′ and the lemma of the preceding section, ω_{ss} is different from zero. The equations (10) therefore have solutions for s and two of the variables u, v, w in terms of the third, passing through the values (s_1, u_0, v_0, w_0) and of class C' near them.

These solutions and the equations (8) *determine a one parameter family of values*

$$\alpha = \alpha(\tau), \qquad \beta = \beta(\tau), \qquad s = \sigma(\tau),$$
$$u = u(\tau), \qquad v = v(\tau), \qquad w = w(\tau)\,*$$

of class C' near $\tau = 0$, which in turn determine a curve

$$(C) \qquad \alpha = \alpha(\tau), \qquad \beta = \beta(\tau)$$

on the surface S and a one parameter family

$$\begin{aligned} x &= \phi[s, u(\tau), v(\tau), w(\tau)] = \phi(s, \tau), \\ y &= \psi[s, u(\tau), v(\tau), w(\tau)] = \psi(s, \tau), \qquad (11) \\ z &= \chi[s, u(\tau), v(\tau), w(\tau)] = \chi(s, \tau), \end{aligned}$$

of the extremals (6). *Through each point of C there passes a unique extremal of the set tangent to the surface S. The functions $\phi(s, \tau)$ and $\phi_s(s, \tau)$ are of class C' near the values $s_0 \gtreqless s \gtreqless s_1$, $\tau = 0$, which define the arc E_{01}, with similar properties for ψ and χ.*

On account of the hypotheses on the function F and the resulting properties of f, it is possible to prove the existence of a one parameter family of extremals on the surface S satisfying the differential equations II or (4), and such that each extremal passes through a point of C, and has the direction determined by the corresponding space extremal of the set (11) tangent to S at that point. Let the length of arc along the surface extremal be denoted by σ and be measured so that $\sigma - \sigma(\tau)$ is the length of arc from C.

There exists, then, a one parameter family of surface extremals with equations on the surface of the form

$$(12) \qquad \alpha = A(\sigma, \tau), \qquad \beta = B(\sigma, \tau),$$

or, in space, of the form

$$\begin{aligned} x &= \xi[A(\sigma, \tau),\ B(\sigma, \tau)] = \xi(\sigma, \tau), \\ y &= \eta[A(\sigma, \tau),\ B(\sigma, \tau)] = \eta(\sigma, \tau), \qquad (13) \\ z &= \zeta[A(\sigma, \tau),\ B(\sigma, \tau)] = \zeta(\sigma, \tau), \end{aligned}$$

* The parameter τ will be $u - u_0$, $v - v_0$ or $w - w_0$.

where the functions $\xi(\sigma, \tau)$ *and* $\xi_\sigma(\sigma, \tau)$ *are of class* C' *near the values*

$$s_1 \leqq \sigma \leqq s_2, \qquad \tau = 0,$$

defining the arc E_{12}. *Along the curve* C *the relations*

$$\phi[\sigma(\tau), \tau] = \xi[\sigma(\tau), \tau], \qquad \phi_s[\sigma(\tau), \tau] = \xi_\sigma[\sigma(\tau), \tau],$$

with similar ones for ψ *and* χ, *are identities in* τ. *They express the fact that at a point of* C *the surface extremal* (12) *and the space extremal* (11) *are tangent to each other.*

Through each point of an extremal on the surface S determined by the equations (12), there passes a space extremal whose direction at the point of intersection coincides with that of the extremal on the surface. This extremal must satisfy the differential equations I. If its length of arc s is measured from the initial value σ at the point of contact with the surface, the existence theorems for differential equations and the properties of the function F justify the following theorem:

THEOREM. *There exists a two parameter family of space extremals tangent to the extremals* (13) *on the surface* S, *and having equations of the form*

$$(14) \qquad x = \Phi(s, \sigma, \tau), \qquad y = \Psi(s, \sigma, \tau), \qquad z = X(s, \sigma, \tau),$$

where Φ *and* Φ_s *are of class* C' *near the values*

$$s = \sigma, \qquad s_1 \leqq \sigma \leqq s_2, \qquad \tau = 0,$$

$$s_2 \leqq s \leqq s_3, \qquad \sigma = s_2, \qquad \tau = 0,$$

defining, respectively, the arcs E_{12} *and* E_{23}. *Similar properties hold for* Ψ *and* X. *The identities*

$$(15) \qquad \Phi(\sigma, \sigma, \tau) \equiv \xi(\sigma, \tau), \qquad \Phi_s(\sigma, \sigma, \tau) \equiv \xi_\sigma(\sigma, \tau),$$

with similar ones for the other functions, express the tangency of (14) *with the surface extremal* (13) *at their point of intersection.*

The identities (15) give rise to a number of further relations important in the following pages. By differentiating the first one and applying the second, it follows that

$$\Phi_\sigma(\sigma, \sigma, \tau) \equiv 0, \qquad \Phi_\tau(\sigma, \sigma, \tau) = \xi_\tau(\sigma, \tau),$$

and by differentiating the second,

$$\Phi_{s\sigma}(\sigma, \sigma, \tau) \equiv \xi_{\sigma\sigma}(\sigma, \tau) - \Phi_{ss}(\sigma, \sigma, \tau) \equiv \bar{x}'' - x''$$

with similar relations for Ψ and X. The determinant

$$\Delta(s, \sigma, \tau) = \begin{vmatrix} \Phi_s & \Phi_\sigma & \Phi_\tau \\ \Psi_s & \Psi_\sigma & \Psi_\tau \\ X_s & X_\sigma & X_\tau \end{vmatrix}$$

has therefore the derivative

$$(16) \qquad \Delta_s(s,\sigma,\tau) = \begin{vmatrix} \Phi_s & \Phi_{s\sigma} & \Phi_\tau \\ \Psi_s & \Psi_{s\sigma} & \Psi_\tau \\ X_s & X_{s\sigma} & X_\tau \end{vmatrix} = \begin{vmatrix} x'' - \bar{x}'' & \xi_\sigma & \xi_\tau \\ y'' - \bar{y}'' & \eta_\sigma & \eta_\tau \\ z'' - \bar{z}'' & \zeta_\sigma & \zeta_\tau \end{vmatrix}$$

for the values (σ, σ, τ) along the surface S. But at a point where the functional determinant of the functions A and B is different from zero, the relations (13) show that this has the value

$$\Delta_s(\sigma,\sigma,\tau) = K\{X(x'' - \bar{x}'') + Y(y'' - \bar{y}'') + Z(z'' - \bar{z}'')\}$$

where K is a factor of proportionality which does not vanish. According to the lemma of § 1, the following result may be stated:

LEMMA 2. *The derivative $\Delta_s(s,\sigma,\tau)$ is different from zero at any point of the curve E_{12} on the surface S at which the determinant*

$$\begin{vmatrix} A_\sigma & A_\tau \\ B_\sigma & B_\tau \end{vmatrix}$$

is different from zero.

3. AUXILIARY THEOREMS

The computations of the following pages can be considerably simplified by the use of some auxiliary theorems, the first of which may be stated as follows:

THEOREM. *If a family of curves*

$$(17) \qquad x = \phi(s,\lambda), \qquad y = \psi(s,\lambda), \qquad z = \chi(s,\lambda)$$

satisfies the relation

$$J^{(x)}\phi_\lambda + J^{(y)}\psi_\lambda + J^{(z)}\chi_\lambda \equiv 0,$$

then the derivative of the integral

$$\int_{s_0}^{s_1} F(\phi,\psi,\chi,\phi_s,\psi_s,\chi_s)\,ds,$$

when s_0, s_1, and λ are functions $s_0(v)$, $s_1(v)$, $\lambda(v)$, is

$$[F_{x'}x_v + F_{y'}y_v + F_{z'}z_v]_0^1.$$

The arguments of the derivatives of F are the same as those in the integral, and

$$(18) \quad x_v = \phi_s s_v + \phi_\lambda \lambda_v, \qquad y_v = \psi_s s_v + \psi_\lambda \lambda_v, \qquad z_v = \chi_s s_v + \chi_\lambda \lambda_v.$$

This theorem follows at once by known methods* provided that the func-

* See M. and B., §2.

tions s_0, s_1, λ are of class C' for $|v| \leqq \epsilon$, and ϕ, ϕ_s of class C' in the region

$$s_0(v) \leqq s \leqq s_1(v), \qquad \lambda = \lambda(v) \qquad\qquad (|v| \leqq \epsilon),$$

with similar properties for ψ and χ.

THEOREM. *If the curves of a second family*

$$(19) \qquad x = \Phi(t, \mu), \qquad y = \Psi(t, \mu), \qquad z = X(t, \mu)$$

adjoin those of the first family for parameter values

$$s = s_1(v), \qquad \lambda = \lambda(v); \qquad t = t_1(v), \qquad \mu = \mu(v),$$

and if corresponding arcs of the two families have the same direction at their common point, then the function

$$(20) \quad G(v) = \int_{s_0}^{s_1} F(\phi, \psi, \chi, \phi_s, \psi_s, \chi_s)\, ds + \int_{t_1}^{t_2} F(\Phi, \Psi, X, \Phi_t, \Psi_t, X_t)\, dt,$$

in which t_2 is a function of v as well as t_1 and μ, has the derivative

$$(21) \qquad G'(v) = [F_{x'} x_v + F_{y'} y_v + F_{z'} z_v]_0^2.$$

The expressions in the bracket must be formed at the point 0 for the first family, and at the point 2 for the second, as described in the first theorem.

The derivative of $G(v)$ has the value

$$[F_{x'} x_v + F_{y'} y_v + F_{z'} z_v]_0^1 + [F_{x'} x_v + F_{y'} y_v + F_{z'} z_v]_1^2,$$

from the first theorem. At the point 1 the two brackets cancel on account of the homogeneity properties of F and the relations

$$\phi[s_1(v), \lambda(v)] = \Phi[t_1(v), \mu(v)],$$

$$\phi_s[s_1(v), \lambda(v)] = \kappa(v)\Phi_t[t_1(v), \mu(v)] \qquad [\kappa(v) > 0],$$

with similar ones for the other functions, which express the intersection and the tangency of the two families. It is understood that the continuity properties of the two families are like those described above for the first.

A similar theorem evidently holds when the number of families is more than two. If the curves of the first family all pass through a fixed point 0 for $s = s_0(v)$, $\lambda = \lambda(v)$, the bracket in (21) vanishes at 0 since the same is true of the derivatives of (17). If the curves of the second family similarly pass through a fixed point 2 for $t = t_2(v)$, $\mu = \mu(v)$, then $G(v)$ is a constant.

The family (19) may be independent of μ, in which case it will be simply an enveloping curve of the family (17). If in this case $t_2(v)$ is a constant, the second of the integrals (20) will be simply the value of J taken along the envelope from 1 to a fixed point 2, and the function $G(v)$ will again be a constant.

4. Jacobi's Conditions for Unilateral Variations

As a result of the assumptions made on the curve E_{03} at the end of §1, it is possible to prove the following further necessary conditions for a minimum:

(VII) *The curve* E_{03} *must not have a point of contact between* 1 *and* 2 *with an enveloping curve* d *of the family* (13) *of extremals on the surface* S.

The proof will be made here for the case where the enveloping curve d does not have a singular point at its point of contact with E_{02}. In that case a stronger condition holds, namely,

(VII′) *The curve* E_{03} *must not have a point of contact with an enveloping curve* d *either between* 1 *and* 2 *or at* 2.

If the curve d has a singular point at its contact point with E_{02}, the condition VII still holds and can be proved by other methods; but the case where d touches E at the point 2 requires, as usual, a special investigation.

The equation which defines d on the surface S is

$$\Delta(\sigma, \tau) = \begin{vmatrix} A_\sigma & A_\tau \\ B_\sigma & B_\tau \end{vmatrix} = 0,$$

where A and B are the functions (12) defining the extremals on the surface. If this is satisfied at a point $(\sigma, \tau) = (\sigma_6, 0)$ of the arc E_{12}, the derivative Δ_σ will be different from zero at that point, as is known from the theory of the plane problem applied to the integral (3). The last equation has therefore a solution

$$\sigma = \Sigma(u), \qquad \tau = \mathrm{T}(u),$$

reducing to $(\sigma_0, 0)$ for $u = 0$ and of class C' near that value. The equations of the envelope in space will be

$$(d) \quad x = \xi[\Sigma(u), \mathrm{T}(u)], \quad y = \eta[\Sigma(u), \mathrm{T}(u)], \quad z = \zeta[\Sigma(u), \mathrm{T}(u)].$$

The parameter u is chosen as one of the values $\pm\tau$, but so as to make the positive direction along d the same as that of the surface extremal tangent to it.

For an arbitrary value of u the corresponding value of τ defines an extremal E_{04} of the set (11) which is tangent to the corresponding extremal E_{45} of the set (13) at a point 4 of the curve C as shown in Fig. 2. Further, E_{45} is tangent to d at a point 5, and d_{56} touches E_{03} at the point 6. The families (11), (13), and d are then of the form described in §3 when $u = v$, and the function

$$G(u) = J(E_{04}) + J(E_{45}) + J(d_{56}) + J(E_{63})$$

is a constant, where E_{63} denotes the arc of E_{03} between 6 and 3. An argument precisely like that used for problems in the plane, shows that d_{56} can not be an extremal on the surface, and since

$$G(0) = J(E_{03})$$

it follows that E_{03} can not minimize J if VII is not satisfied.

If the condition VII′ is satisfied together with those postulated at the end of §1, a final necessary condition is the following:

(VIII) *The curve* E_{03} *must not have a point of contact between* 2 *and* 3 *with an enveloping surface D of the family* (14) *of extremals tangent to S*.

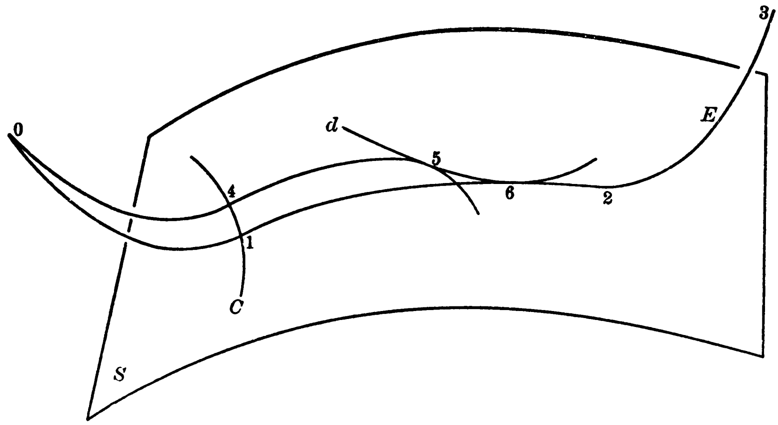

FIG. 2.

Here again the proof will be made for the case where D does not have a singular point at its contact with E_{03}, and the stronger condition holding under those circumstances is

(VIII′) *The curve* E_{03} *must not have a point of contact with D either between* 2 *and* 3 *or at* 3.

The points on the enveloping surface are defined by the equation

$$\Delta(s, \sigma, \tau) = \begin{vmatrix} \Phi_s & \Phi_\sigma & \Phi_\tau \\ \Psi_s & \Psi_\sigma & \Psi_\tau \\ X_s & X_\sigma & X_\tau \end{vmatrix} = 0. \tag{22}$$

On account of Lemma 2 of §2, the determinant is different from zero on the arc E_{23} near the point 2, though it vanishes at 2 itself. If the equation is satisfied at a point 7 of the arc E_{23} for which $(s, \sigma, \tau) = (s_7, s_2, 0)$, and further defines a non-singular enveloping surface D, then the argument of Mason and Bliss* shows that there is a curve on D,

$$x = x(u), \quad y = y(u), \quad z = z(u), \tag{d}$$

passing through the point 7 for $u = 0$, and which is tangent to a one parameter family of the extremals (15) at points defined by equations of the form

$$s = S(u), \quad \sigma = \Sigma(u), \quad \tau = \mathrm{T}(u),$$

* See M. and B., §3.

where S, Σ, T reduce to $s_7, s_2, 0$ for $u = 0$. The parameter u can be so chosen that the positive direction on the curve d coincides with the positive direction of the extremal (15) at the point defined by these functions.

An arbitrarily chosen value u defines an extremal E_{04} of the set (11) tangent to an extremal E_{45} of the set (12) at a point 4 of the curve C; and E_{45} is in turn tangent at a point 5 to the extremal E_{56} defined by the equations

$$(23) \qquad x = \Phi(s, \Sigma, \mathrm{T}), \qquad y = \Psi(s, \Sigma, \mathrm{T}), \qquad z = \mathrm{X}(s, \Sigma, \mathrm{T}).$$

Finally this last curve is tangent to d_{67} at a point 6. When u is set equal to v, the families (11), (13), (23), and d_{67} are of the kind described in §3, and the function

$$G(u) = J(E_{04}) + J(E_{45}) + J(E_{56}) + J(d_{67}) + J(E_{73})$$

is a constant. Since

$$G(0) = J(E_{03})$$

and d can not be an extremal, it follows that E_{03} can not minimize J if VIII is not satisfied.

5. The Existence of a Field

If the arc E_{03} satisfies the stronger set of eight conditions described in §§1, 4, it will surely minimize the integral J with respect to variations sufficiently near to it and lying on one side of the surface S. But before proceeding to the proof of this fact it is necessary to show how a field may be constructed about the minimizing arc.

For this purpose consider first the family (14) which contains the arcs E_{12} and E_{23}, respectively, for the parameter values

$$(\Pi) \qquad \begin{aligned} &s = \sigma, \qquad s_1 \leqq \sigma \leqq s_2, \qquad \tau = 0; \\ &s_2 \leqq s \leqq s_3, \qquad \sigma = s_2, \qquad \tau = 0. \end{aligned}$$

The symbol Π_ϵ will be used to denote the set of points (s, σ, τ) which satisfy either of the following sets of conditions:

$$(\Pi_\epsilon) \qquad \begin{aligned} &\sigma \leqq s \leqq \sigma + \epsilon, \qquad \sigma(\tau) \leqq \sigma \leqq s_2 + \epsilon, \qquad |\tau| \leqq \epsilon; \\ &\sigma + \epsilon \leqq s \leqq s_3 + \epsilon, \qquad |\sigma - s_2| \leqq \epsilon, \qquad |\tau| \leqq \epsilon. \end{aligned}$$

If the constant ϵ is chosen sufficiently small, any two distinct points of Π_ϵ will be transformed by the equations (14) *into distinct points (x, y, z); and in the interior of Π_ϵ the determinant* (22) *will be everywhere different from zero.*

To prove this it will first be shown that in a suitably chosen neighborhood of a point $\pi_0(s_0, \sigma_0, \tau_0)$ satisfying the second of the conditions (II), no two distinct points $\pi(s, \sigma, \tau)$, $\pi'(s', \sigma', \tau')$ of the region Π_ϵ can define the same point (x, y, z) by means of the equations (14). By Taylor's formula,*

$$(24) \quad \Phi(s', \sigma', \tau') - \Phi(s, \sigma, \tau) = (s' - s)A_1 + (\sigma' - \sigma)A_2 + (\tau' - \tau)A_3$$

where, for example,

$$(25) \qquad A_2 = \int_0^1 \Phi_\sigma(s'', \sigma'', \tau'')\,du,$$

$s'' = s + u(s' - s), \qquad \sigma'' = \sigma + u(\sigma' - \sigma), \qquad \tau'' = \tau + u(\tau'' - \tau),$ and the other coefficients have similar values. The expression (24) and the similar ones for Ψ and X can vanish simultaneously only when the determinant of the coefficients vanishes. But this is impossible near π_0 since the determinant (22) is different from zero at π_0 and the derivatives of Φ, Ψ, X are continuous.

Near a point π_0 satisfying the first of the conditions II the expression (25) for A_2 can be put in the form

$$A_2 = \int_0^{'} (s'' - \sigma'') \int_0^{'} \Phi_{s\sigma}[\sigma'' + v(s'' - \sigma''), \sigma'', \tau'']\,dv\,du,$$

since Φ_σ vanishes when its first two arguments are the same. The point (s'', σ'', τ'') is always in Π_ϵ when π and π' are in Π_ϵ and $0 \leqq u \leqq 1$, and hence $s'' - \sigma''$ is never negative. By a double application of the mean value theorem for a definite integral

$$A_2 = (s_1 - \sigma_1)\Phi_{s\sigma},$$

where the arguments of $\Phi_{s\sigma}$ contain properly chosen intermediate values for u and v, and (s_1, σ_1, τ_1) is the middle point of the segment $\pi\pi'$.

Near a point π_0 satisfying the second of the conditions (II) the difference (24) can therefore be put into the form

$$(26) \qquad (s' - s)A_1 + (\sigma' - \sigma)(s_1 - \sigma_1)A_2' + (\tau' - \tau)A_3,$$

where A_1, A_2', A_3 approach continuously the values of Φ_s, $\Phi_{s\sigma}$, Φ_τ at the point π_0 when π and π' approach π_0. This expression and the similar ones for Ψ and X can vanish simultaneously, for π and π' sufficiently near to π_0, only when

$$s' - s = \tau' - \tau = (\sigma' - \sigma)(s_1 - \sigma_1) = 0,$$

since at the point π_0 the determinant (22) is different from zero. But if $s' = s$ and $\tau' = \tau$, the values of σ' and σ must be distinct since π and π' are distinct. The difference $s_1 - \sigma_1$ can not vanish since $s_1 = s = s'$ is greater than the larger of σ and σ' on account of the conditions (Π_ϵ). Hence near π_0 the

* See Jordan, *Cours d'Analyse*, 2d ed., vol. 1, p. 247.

difference (24) and the similar ones for Ψ and X can not all vanish if π and π' are distinct.

Suppose now that it should not be possible to choose a constant ϵ such as is described in the first part of the italicized statement, and consider a sequence $\{\epsilon_k\}$ of decreasing positive constants having zero as its limit. For any choice of k there would exist at least one pair of distinct points,

$$(27) \qquad (s, \sigma, \tau)_k, \qquad (s', \sigma', \tau')_k$$

in the region Π_{ϵ_k} and defining the same point $(x, y, z)_k$ by means of the equations (14). The two sequences (27) would have points of condensation

$$(28) \qquad (s_0, \sigma_0, \tau_0), \qquad (s_0', \sigma_0', \tau_0')$$

necessarily lying in the region Π, and these two points must be the same. Otherwise they would define distinct points of the arc E_{13}, and on account of the continuity of the functions (14) the pairs of values (27) which lie sufficiently near to the values (28) could not define the same point $(x, y, z)_k$. Neither could the pair of sets of values (27) define the same point $(x, y, z)_k$ in case the two condensation points (28) were identical, as has been seen in the paragraphs just preceding.

A similar argument shows that ϵ can be chosen so as to satisfy the last part of the italicized statement. Otherwise theie would be a sequence $(s, \sigma, \tau)_k$ of interior points of the respective regions Π_{ϵ_k} at each of which the determinant (22) vanishes. Such a sequence must have a condensation point π_0 in Π. But π_0 could not satisfy the second of the conditions (Π), since at such points $\Delta(s, \sigma, \tau)$ is different from zero. Neither could it satisfy the first condition; for near such a point π_0 the value of $\Delta_s(s, \sigma, \tau)$ is different from zero, on account of Lemma 2 and because for $s > \sigma$

$$\Delta(s, \sigma, \tau) = (s - \sigma)\Delta_s[s + \theta(s - \sigma), \sigma, \tau] \neq 0.$$

The points of the sequence $(s, \sigma, \tau)_k$ which condense on π_0 have $s > \sigma$, since they are interior points, and hence the existence of the sequence is contradicted.

If the region Π_ϵ *is chosen as described above, its image in the xyz-space, after transformation by equations* (14), *is a region* $\mathfrak{F}_2$, *a continuum and its boundary, whose boundary points are the images of the boundary points of* Π_ϵ. *The correspondence between* $\mathfrak{F}_2$ *and* Π_ϵ *is one-to-one. The single valued functions*

$$(29) \qquad s = s(x, y, z), \qquad \sigma = \sigma(x, y, z), \qquad \tau = \tau(x, y, z)$$

so defined are continuous over $\mathfrak{F}_2$, *and in the interior of* $\mathfrak{F}_2$ *they have continuous derivatives.*

The first part of this statement follows at once from some theorems proved by Bliss,* since the images of the interior and boundary of Π_ϵ are distinct. The preceding paragraphs of this paper show that the correspondence between $\mathfrak{F}_2$ and Π_ϵ is one-to-one. An interior point of $\mathfrak{F}_2$ corresponds to an interior point of Π_ϵ at which $\Delta(s, \sigma, \tau) \neq 0$, and the fundamental theorem of the implicit function theory shows at once that the functions (29) are of class C' at such points. If a sequence of points $(x, y, z)_k$ of $\mathfrak{F}_2$ approaches a boundary point (x', y', z') of $\mathfrak{F}_2$, the corresponding values $(s, \sigma, \tau)_k$ defined by the equations (29) must approach the values (s', σ', τ') which define (x', y', z') by means of equations (14), as may readily be seen from the continuity of the functions Φ, Ψ, X. Hence the functions (29) are continuous everywhere in $\mathfrak{F}_2$.

There exists a field $\mathfrak{F}_1$ about the arc E_{01} through each point of which there passes a unique extremal of the set (6) *defined by equations of the form*

(30)
$$s = s(x, y, z), \qquad u = u(x, y, z),$$
$$v = v(x, y, z), \qquad w = w(x, y, z).$$

These functions are continuous everywhere in $\mathfrak{F}_1$ and have continuous derivatives except at the point 0.†

The symbol $\mathfrak{F}_{12}$ will be used to denote the set of points (x, y, z) consisting of the interior of $\mathfrak{F}_2$, the points of S in $\mathfrak{F}_2$, and the points interior to $\mathfrak{F}_1$. Let V be a continuous curve

$$(V) \qquad x = \bar{x}(t), \qquad y = \bar{y}(t), \qquad z = \bar{z}(t) \qquad (t_0 \leqq t \leqq t_3)$$

consisting of a finite number of non-singular arcs of class C', joining the points 0 and 3, and lying in the region $\mathfrak{F}_{12}$ on the same side of S with the arc E_{01}.

If $\mathfrak{F}_1$ is taken sufficiently small the only points where V can enter $\mathfrak{F}_2$ are on the boundary B of $\mathfrak{F}_2$ which corresponds to the parameter values

$$(31) \qquad \sigma \leqq s < \sigma + \epsilon, \qquad \sigma = s(t), \qquad |\tau| < \epsilon.$$

A neighborhood of the point 1 can be chosen so small that the only boundary points of $\mathfrak{F}_2$ in it are points of the surface S and points on the boundary B defined by the conditions (31). Further, $\mathfrak{F}_1$ can be taken so small that the only points common to it and $\mathfrak{F}_2$ are in the same neighborhood. Under these circumstances the curve V can enter $\mathfrak{F}_2$ only at points of B or at points of S

* *The Princeton Colloquium Lectures on Mathematics*, p. 38.

† For the existence of the field in the neighborhood of the point 0, see Bliss and Mason, *Fields of extremals in space*, these Transactions, vol. 11 (1910), p. 328; also references at the beginning of the same paper; also Szücs, Mathematische Annalen, vol. 71 (1911), p. 380. The existence of the field about the rest of the arc follows by the usual methods. A proof analogous to that just preceding in the text is more elegant than any o these.

distinct from B. The latter are, however, excluded since V lies entirely on one side of S near the point 1.

6. Sufficient Conditions for a Minimum

It is proposed to prove in this section that an arc E_{03} of the kind described at the end of §5, will give the integral J a value at least equal to $J(E_{03})$. To each value of t defining a point 6 of V_{03} in $\mathcal{F}_2$, there will correspond a unique curve

$$E_{06} = E_{04} + E_{45} + E_{56},$$

shown in Fig. 1, where E_{04}, E_{45}, E_{56} belong respectively to the families (11), (13), and (14); while if t defines a point 6 which does not lie in $\mathcal{F}_2$ it will determine a unique extremal E_{06} of the family (6).

The integral $J(E_{06})$, is a continuous function of t which has the derivative

$$\frac{d}{dt} J(E_{06}) = F_{x'}\,\bar{x}' + F_{y'}\,\bar{y}' + F_{z'}\,\bar{z}'\,\Big|^{6}. * \tag{32}$$

In this expression the arguments of the derivatives of F are the values (x, y, z, x', y', z') which define the point and direction of E_{06} at the point 6, while $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ are the derivatives at the same point of the functions (V) defining V.

To prove this consider the one parameter family of curves E_{06} determined by the points of V_{03} interior to $\mathcal{F}_2$. It is determined by substituting the functions defining V in the expressions (29) for s, σ, τ in terms of x, y, z. The functions $s(t)$, $\sigma(t)$, $\tau(t)$ so determined are of class C' near a value of t defining an interior point of $\mathcal{F}_2$ unless t is one of the exceptional values at which V has a corner point. By substituting $\sigma(t)$, $\tau(t)$ in the equations (11), (13), and (14), the three families E_{04}, E_{45}, E_{56} are determined with properties similar to those of the auxiliary theorems of §3. It follows readily that the derivative of $J(E_{06})$ has the value given in the theorem.

The argument is the same for a point 6 in $\mathcal{F}_1$ but not in $\mathcal{F}_2$, except that the equations whose solutions determine the family E_{06} are then

$$x(t) = \phi(s, u, v, w), \qquad y(t) = \psi(s, u, v, w), \qquad z(t) = \chi(s, u, v, w),$$
$$u^2 + v^2 + w^2 = 1.$$

When the point 6 lies on the surface S at the point 0, or on the boundary B, it is more difficult to show the existence of the derivative. But the desired result can be obtained by the use of expansions of the functions (14) similar to (26), by means of the properties of the functions (6) at the point 0 described in the paper referred to in the preceding section, and by special considerations at a point of B.

* At corner points of the curve V this should be thought of as a forward derivative.

If the stronger set of eight properties described in the preceding sections is satisfied by the arc E_{03}, then

$$J(E_{03}) < J(V_{03}) \tag{33}$$

for any curve V_{03} distinct from E_{03} in the field $\mathfrak{F}_{12}$ and on the same side of S with the arc E_{03}. It is understood that V_{03} consists of a finite number of non-singular arcs of class C'.

For consider the continuous function

$$W(t) = J(E_{06}) + J(V_{63}),$$

where 6 is the point of V determined by the parameter value t. With the help of the last theorem but one, the derivative of $W(t)$ exists except at a finite number of values of t, and has itself the value

$$W'(t) = -E(x, y, z;\ x', y', z';\ \bar{x}', \bar{y}', \bar{z}'). \tag{34}$$

On account of IV and the properties of the function E this is always negative except when the positive directions along V_{63} and E_{06} coincide at the point 6. Furthermore

$$W(t_0) = J(V_{03}), \qquad W(t_1) = J(E_{03}).$$

The inequality (33) must therefore be satisfied unless the positive directions along E_{06} and V_{63} coincide at every point 6 of V where the derivative (34) is well defined.

If the positive directions along V coincide with those of E_{06} at every point of V, then V_{03} and E_{03} must be coincident. The proof is somewhat complicated but may be made with the help of the following lemma:

LEMMA 3. *Any arc of V defined by an interval $t' \leqq t \leqq t''$ and containing no points of S or B, is an extremal of one of the families* (6), (14) *along which s increases from s' to s'' as t traverses the interval $t'\,t''$. An arc of V lying entirely on the surface S must be an arc of one of the surface extremals* (13) *with similar properties.*

An arc of V with which the lemma is concerned must lie in the interior of $\mathfrak{F}_1$ or $\mathfrak{F}_2$, or else on the surface S. The proofs in the three cases are quite similar, so that it will be sufficient to suppose the arc V to lie entirely on S. The values $s = \sigma, \sigma, \tau$ are then continuous functions $\sigma(t)$, $\tau(t)$ defined over the interval $t' \leqq t \leqq t''$ by equations (V) and (29), and they satisfy the equations

$$\begin{gathered} x(t) = \xi[\sigma(t), \tau(t)], \qquad y(t) = \eta[\sigma(t), \tau(t)], \\ z(t) = \zeta[\sigma(t), \tau(t)]. \end{gathered} \tag{35}$$

At least one of the determinants of the matrix

$$\begin{Vmatrix} \xi_\sigma & \eta_\sigma & \zeta_\sigma \\ \xi_\tau & \eta_\tau & \zeta_\tau \end{Vmatrix}$$

is different from zero at every value of t on account of VII′ and the relations (13). Hence the theory of implicit functions applied to a suitably chosen pair of the equations (35) shows that $\sigma(t)$ and $\tau(t)$ are of class C'. If the direction of V coincides with that of the surface extremal at each point of the arc in question, then

$$x'(t) = \kappa\,\xi_\sigma, \qquad y'(t) = \kappa\,\eta_\sigma, \qquad z'(t) = \kappa\,\zeta_\sigma \qquad (\kappa > 0),$$

and by differentiating equations (35) the relations

$$\sigma'(t) = \kappa, \qquad \tau'(t) = 0$$

are readily deduced. Hence the lemma is true for an arc on S, and for the other two cases by similar arguments.

LEMMA 4. *If t_4 is a lower bound of values of t such that for $t_4 \leqq t \leqq t_3$ the curve V_{03} coincides with E_{03}, then no value $t' < t_4$ can define a point of V interior to $\mathfrak{F}_2$, or a point of V on the surface S and distinct from B.*

There must be a lower bound t_4 of the kind described since near the point 3 the curve V_{03} coincides with an extremal of the field passing through the point 3, according to Lemma 3, and E_{03} is the only such extremal.

If t' defined a point interior to $\mathfrak{F}_2$, then there would be a first value $t'' > t'$ such that the arc $t'\,t''$ of V would be entirely interior to $\mathfrak{F}_2$ and distinct from E_{03} except at the value t''. The value t'' would necessarily define a point either of B, or S, or E_{23}. But the arc $t'\,t''$ of V would in that case be an extremal arc along which s increases with t from a value s' to a value s''. This is, however, impossible, since any such arc which is distinct from B, S, E_{23} for $s = s'$ retains this property for all values $s > s'$.

With the help of this result and by similar arguments it follows that t' could not define a point of S distinct from B.

The identity of V_{03} and E_{03} now follows without difficulty. The value $t = t_4$ must in fact define a point 4 on the arc E_{01}, since in case this is not true there would certainly be values $t' < t_4$ satisfying the conditions of Lemma 4. The arc $t_0\,t_1$ of V then lies entirely in the field $\mathfrak{F}_1$, and no point of it can be distinct from E_{01}. Otherwise there would be an extremal arc of the field $\mathfrak{F}_1$ distinct from E_{01} whose end point, as s increased toward the parameter value defining it, would lie on E_{01}. There is, however, no such arc.